BOUNDING THE PRICE OF ANARCHY FOR GAMES WITH PLAYER-SPECIFIC COST FUNCTIONS

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ABSTRACT. We study the efficiency of equilibria in atomic splittable congestion games on networks. We consider the general case where players are not affected in the same way by the congestion. Extending a result by Cominetti, Correa, and Stier-Moses (*The impact of oligopolistic competition in networks*, Oper. Res., **57**, 1421–1437 (2009)), we prove a general bound on the price of anarchy for games with player-specific cost functions. This bound generalizes some of their results, especially the bound they obtain for the affine case. However our bound still requires some dependence between the cost functions of the players. In the general case, we prove that the price of anarchy is unbounded, by exhibiting an example with affine cost functions and only two players.

1. Introduction

1.1. Motivation.

In many economic fields, companies share common resources while being non coordinated. These resources are often owned by agents that are paid for the access they provide to these companies. These resources are for instance machines in a flexible manufacturing environment, means of transportation in a freight context, or arcs in a telecommunication network. An increase of demand for a resource often leads to an increase of its cost, because the fees increase, or because delay is created. This increase of the cost is seen as a congestion on the resource.

Taking a game-theoretical point of view, Cominetti, Correa, and Stier-Moses [3] studied the efficiency loss implied by the lack of coordination. The companies become *players* in a *congestion game* and the resources become arcs of a network. If there is no conditional use between the resources, the parallel-link network often models correctly the interaction between the players, see for instance Orda, Rom, and Shimkin [9], Hayrapetyan, Tardos, and Wexler [7], Wan [16]. However, in many situations, several resources have to be chosen simultaneously by each company, but not all subsets of resources are possible. Modeling the possible subsets as routes in a network is a way to cover these situations and makes sense in a freight transportation context for instance. All possible congestion games cannot be modeled with a network, but this representation is helpful and the results often extend to general congestion games without additional work (this is the case for the game we deal with in the present work). To model the congestion, each arc is endowed with a cost function.

A useful notion to quantify the loss of efficiency is the *price of anarchy*, introduced by Koutsoupias and Papadimitriou [8]. The price of anarchy is the worst-case ratio between the social cost at equilibrium and the best possible social cost. Its interest has lead to a considerable amount of work since the seminal paper of Roughgarden and Tardos [14]. Cominetti, Correa, and Stier-Moses [3] are able to prove general upper bounds for the price of anarchy of atomic splittable games, which are valid in a large set of situations. These results have been extended by Harks [6] and Roughgarden and Schoppmann [13].

One of the main assumptions in these papers is that, for each arc of the network, the players all have the same cost function. However, depending on their size, the nature of the goods they carry, or other features, the companies are not equally affected by the congestion. Thus, allowing

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the companies to have their own cost functions makes sense. To the best of our knowledge, this assumption has not been relaxed in the context of the computation of the price of anarchy, except in Gairing, Monien, and Tiemann [5] for different models (atomic unsplittable and nonatomic games). In this paper, we allow player-specific cost functions and extend some of the results of Cominetti, Correa, and Stier-Moses [3] in this more general setting.

2. Model and main results

2.1. **Model.** The description of the game we deal with in this paper goes as follows. We are given a directed network $D = (V, A)$ and K players identified with the integers $1, \ldots, K$. Each player $k \in [K]$ has to send d^k units of flow, its demand, from an origin $s^k \in V$ to a destination $t^k \in V$ in this network. A strategy for player k is therefore an s^k-t^k flow $\boldsymbol{x}^k \in \mathbb{R}_+^A$ of value d^k. Such a flow is an element of

$$\mathcal{F}^k = \left\{ \boldsymbol{y} \in \mathbb{R}_+^A : \sum_{a \in \delta^+(s^k)} y_a - \sum_{a \in \delta^-(s^k)} y_a = d^k \quad \text{and} \right.$$

$$\left. \sum_{a \in \delta^+(v)} y_a = \sum_{a \in \delta^-(v)} y_a, \quad \forall v \in V \setminus \{s^k, t^k\} \right\}$$

and is referred as a *feasible flow for player k*. Throughout the paper, we consider the decisions of the players as a multiflow $\vec{\boldsymbol{x}} = (\boldsymbol{x}^1, \ldots, \boldsymbol{x}^K) \in \mathbb{R}_+^{A \times K}$. It is *feasible* if it is an element of $\mathcal{F}^1 \times \cdots \times \mathcal{F}^K$.

Each player k has his own vector of cost functions $\boldsymbol{c}^k = (c_a^k)_{a \in A}$ where for each arc a the cost function $c_a^k(\cdot)$ is a $\mathbb{R}_+ \to \mathbb{R}_+$ function. We assume given a set of *allowable cost functions* $\mathcal{C}$ in which $\vec{\boldsymbol{c}} = (\boldsymbol{c}^1, \ldots, \boldsymbol{c}^K)$ is taken. Since each player has his own vector of cost functions, the game is said to be with player-specific cost functions. We will sometime say that it is a *multiclass* game. On the contrary, if $\boldsymbol{c}^1 = \cdots = \boldsymbol{c}^K$ for every $\vec{\boldsymbol{c}} \in \mathcal{C}$, we speak of a *single-class* game.

We denote the total flow on an arc a by $x_a = \sum_{k \in [K]} x_a^k$. The cost experienced by a player k is

$$C^k(\vec{\boldsymbol{x}}) = \sum_{a \in A} x_a^k c_a^k(x_a).$$

The goal of this player consists in sending its d^k units of flows while minimizing this cost.

A feasible multiflow $\vec{\boldsymbol{x}}^{NE} = (\boldsymbol{x}^{NE,1}, \ldots, \boldsymbol{x}^{NE,K})$ is a *Nash equilibrium* if for each player k, we have

$$C^k(\vec{\boldsymbol{x}}^{NE}) = \min_{\boldsymbol{y} \in \mathcal{F}^k} C^k(\boldsymbol{y}, \vec{\boldsymbol{x}}^{NE,-k}), \tag{1}$$

where $(\boldsymbol{y}, \vec{\boldsymbol{x}}^{NE,-k}) = (\boldsymbol{x}^{NE,1}, \ldots, \boldsymbol{x}^{NE,k-1}, \boldsymbol{y}, \boldsymbol{x}^{NE,k+1}, \ldots, \boldsymbol{x}^{NE,K})$.

This kind of games belongs to the class of *atomic splittable network congestion games.* Atomic, because each player has nonnegligible impact. Splittable, because the goods are seen as a flow whose support is not necessarily a unique route. For the game we consider in this paper, a Nash equilibrium always exists [11, 9]. However, it may not be unique and even the total flows on the arcs may vary among the multiple Nash equilibria [10, 2].

The *social cost* of a multiflow is defined as

$$C(\vec{\boldsymbol{x}}) = \sum_{k \in [K]} C^k(\vec{\boldsymbol{x}}) = \sum_{k \in [K]} \sum_{a \in A} x_a^k c_a^k(x_a).$$

A multiflow of minimal social cost is a *social optimum.*

2.2. **Main results.** An instance of the game is defined by the network, the set of players with their origin-destination pairs, their demands, and their cost functions. We denote by $\mathrm{NE}(I)$ the set of Nash equilibria for an instance I and by $\vec{\boldsymbol{x}}^{OPT}(I)$ a feasible multiflow achieving the minimal social cost.

This paper is focused on the *price of anarchy* (PoA) of the game. Given a set of allowable instances $\mathcal{I}$, the price of anarchy is

$$\mathrm{PoA} = \sup_{I \in \mathcal{I}} \sup_{\vec{\boldsymbol{x}} \in \mathrm{NE}(I)} \frac{C(\vec{\boldsymbol{x}})}{C(\vec{\boldsymbol{x}}^{OPT}(I))}.$$

Theorem 1. *Consider an atomic splittable network congestion game with player-specific cost functions. Suppose that the available cost functions in* $\mathcal{C}$ *are differentiable, nonnegative, increasing, and convex, and define* $\lambda(\mathcal{C}) = \sup_{a \in A,\, k,\ell \in [K],\, \vec{\boldsymbol{c}} \in \mathcal{C},\, x \in \mathbb{R}_+} \dfrac{x(c_a^\ell)'(x)}{c_a^k(x)}$.
If $\lambda(\mathcal{C}) < 3$*, we have*

$$\mathrm{PoA} \leq \frac{1}{1 - \lambda(\mathcal{C})/3}.$$

This theorem is also valid if we are in the single-class case, in which case $\lambda(\mathcal{C})$ coincides with $\gamma(\mathcal{C}) - 1$ in Cominetti et al. [3].

The upper bound on the price of anarchy given in Theorem 1 can be improved with a more complicated formula, see Proposition 5 below. A corollary is then the following proposition, which is the generalization of Proposition 3.5 in Cominetti et al. [3] to the multiclass case.

Proposition 1. *Consider an atomic splittable network congestion game with player-specific cost functions. Suppose that the allowable cost functions are affine of the form* $c_a^k(x) = p_a^k x + q_a^k$ *with* $p_a^k > 0, q_a^k \geq 0$*, and define* $\Delta = \sup_{a \in A} \dfrac{\sup_{\ell \in [K]} p_a^\ell}{\inf_{k \in [K]} p_a^k}$.
If $\Delta < 3$*, we have*

$$\mathrm{PoA} \leq \frac{3\Delta(K-1) + 4}{\Delta(3 - \Delta)(K-1) + 4}.$$

Proposition 1 uses the parameter Δ with the same definition and with a similar purpose as in Gairing et al. [5]. This parameter is independent of the constant terms of the functions. This bound is valid for $\Delta < 3$, i.e. when the marginal cost caused by the congestion does not vary too much among the players. In particular, it handles cases where the marginal impact of the congestion for any player is at most twice the one of the others. In practice, for example in the transportation context, these situations are likely to happen. Note that in the single-class case, we have $\Delta = 1$, and our proposition coincides with the aforementioned Proposition 3.5 of Cominetti et al. [3] giving a bound of $\frac{3K+1}{2K+2}$.

If the cost functions are affine, Proposition 1 shows that the price of anarchy is bounded when $\Delta < 3$. However, according to the next proposition, it is unbounded in general, even when the cost functions are affine.

Proposition 2. *For any* $M > 0$*, there is an instance of an atomic splittable network congestion game with player-specific affine cost functions, with two players, and with*

$$\mathrm{PoA} > M.$$

This result contrasts with the single-class case. Indeed, in this case it has been showed that the price of anarchy is bounded for polynomial cost functions of degree d: Harks [6], followed by

Roughgarden and Schoppmann [13] found the closed-form upper bound of $\left(\frac{1+\sqrt{d+1}}{2}\right)^{d+1}$. Bounds also exist for different games that are nonatomic games [15] and atomic unsplittable games [1].

The remaining of the paper is organized as follows. Section 3 gives a general but not explicit bound on the price of anarchy, from which we deduce Theorem 1 and Proposition 1. Section 4 presents the proof of Proposition 2. In Section 5 we discuss the results and some open questions.

3. The price of anarchy for general cost functions

3.1. **Preliminary remarks.** Throughout the paper, the components of an $\boldsymbol{x} \in \mathbb{R}_+^K$ are denoted x^k. Given a K-tuple of cost functions $\boldsymbol{c} = (c^1, \ldots, c^K)$, we define $\widetilde{c}^k : \mathbb{R}_+^K \to \mathbb{R}_+$ by

$$\widetilde{c}^k(\boldsymbol{x}) = \frac{\partial}{\partial x^k}\left(x^k c^k(x)\right) \qquad \text{where } x = \sum_{\ell \in [K]} x^\ell.$$

We have then

$$\widetilde{c}^k(\boldsymbol{x}) = x^k c^{k\prime}(x) + c^k(x).$$

Note that the c^k in Cominetti et al. [3] coincides with our $\widetilde{c}^k$.

The following proposition is standard in this context and is obtained by writing the optimality conditions of Equation (1).

Proposition 3. *The multiflow $\vec{\boldsymbol{x}}^{NE}$ is a Nash equilibrium if and only if, for all $k \in [K]$, it satisfies*

$$\sum_{a \in A} \widetilde{c}_a^k(\boldsymbol{x}_a^{NE})(y_a^k - x_a^{NE,k}) \geq 0, \quad \textit{for any feasible flow } \boldsymbol{y}^k \textit{ for player } k, \tag{2}$$

where $\boldsymbol{x}_a^{NE} = (x_a^{NE,1}, \ldots, x_a^{NE,K}) \in \mathbb{R}_+^K$.

3.2. **A general bound.** Following Cominetti et al. [3], for a K-tuple of cost functions $\boldsymbol{c} = (c^1, \ldots, c^K)$, we define

$$\beta(\boldsymbol{c}) = \sup_{\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}_+^K} \frac{\sum_{k \in [K]} \left[\left(\widetilde{c}^k(\boldsymbol{x}) - c^k(y)\right) y^k + \left(c^k(x) - \widetilde{c}^k(\boldsymbol{x})\right) x^k\right]}{\sum_{k \in [K]} x^k c^k(x)},$$

where x stands for $\sum_{k \in [K]} x^k$ and y for $\sum_{k \in [K]} y^k$. We assume $0/0 = 0$. Note that $\beta(\boldsymbol{c}) \geq 0$, since the function we want to maximize is zero when $\boldsymbol{x} = \boldsymbol{y}$. We also define $\beta(\mathcal{C}) = \sup_{\vec{\boldsymbol{c}} \in \mathcal{C},\, a \in A} \beta(\boldsymbol{c}_a)$, where $\boldsymbol{c}_a = (c_a^1, \ldots, c_a^K)$.

The following proposition gives a general bound, yet nonexplicit, on the price of anarchy. For sake of simplicity, we assume that $(1 - \beta(\mathcal{C}))^{-1} = +\infty$ for $\beta(\mathcal{C}) \geq 1$.

Proposition 4. *Let $\vec{\boldsymbol{x}}^{NE}$ be a Nash equilibrium and $\vec{\boldsymbol{x}}^{OPT}$ be a social optimum. Then*

$$C(\vec{\boldsymbol{x}}^{NE}) \leq \frac{1}{1 - \beta(\mathcal{C})} C(\vec{\boldsymbol{x}}^{OPT}).$$

This proposition coincides with Proposition 3.2 of Cominetti et al. [3] in the single-class case. They refer to Roughgarden [12] and use ideas of Correa et al. [4]. The proof is routine.

Proof of Proposition 4. We have

$$\begin{aligned}
C(\vec{\boldsymbol{x}}^{NE}) &= \sum_{k\in[K]}\sum_{a\in A}\left(c_a^k(x_a^{NE}) - \tilde{c}_a^k(\boldsymbol{x}_a^{NE})\right)x_a^{NE,k} + \tilde{c}_a^k(\boldsymbol{x}_a^{NE})x_a^{NE,k} \\
&\leq \sum_{k\in[K]}\sum_{a\in A}\left(c_a^k(x_a^{NE}) - \tilde{c}_a^k(\boldsymbol{x}_a^{NE})\right)x_a^{NE,k} + \tilde{c}_a^k(\boldsymbol{x}_a^{NE})y_a^k \\
&\leq \sum_{a\in A}\left[\beta(\boldsymbol{c}_a)\sum_{k\in[K]} x_a^{NE,k}c_a^k(x_a^{NE})\right] + C(\vec{\boldsymbol{y}}) \\
&\leq \beta(\mathcal{C})C(\vec{\boldsymbol{x}}^{NE}) + C(\vec{\boldsymbol{y}})
\end{aligned}$$

where we use Equation (2) to get the first inequality and the definition of $\beta(\cdot)$ to get the second inequality.

We finish by taking $\vec{\boldsymbol{y}} = \vec{\boldsymbol{x}}^{OPT}$. □

3.3. **Computation of the bound.** We give now an explicit upper bound on $\beta(\boldsymbol{c})$.

Proposition 5. *Consider a K-tuple of cost functions $\boldsymbol{c} = (c^1,\ldots,c^K)$ and define*

$$\delta(\boldsymbol{c}) = \sup_{k,\ell\in[K],\,x\in\mathbb{R}_+}\frac{(c^\ell)'(x)}{(c^k)'(x)} \quad \textit{and} \quad \lambda(\boldsymbol{c}) = \sup_{k,\ell\in[K],\,x\in\mathbb{R}_+}\frac{x(c^\ell)'(x)}{c^k(x)}.$$

If each c^k is differentiable, nonnegative, increasing, and convex, then the following inequality holds

$$\beta(\boldsymbol{c}) \leq \frac{\lambda(\boldsymbol{c})}{3}\frac{1}{1+\frac{4}{3}\frac{1}{\delta(\boldsymbol{c})(K-1)}}. \tag{3}$$

Before proving this proposition, let us note that the special case $K = 1$ gives $\beta(\boldsymbol{c}) = 0$ and a price of anarchy of 1 as expected: if there is only one player, Nash equilibrium and social optimum coincide.

Proof of Proposition 5. Here and throughout the proof, $c^{k\prime}$ stands for the derivative of c^k. We have thus

$$\beta(\boldsymbol{c}) = \sup_{\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}_+^K}\frac{f(\boldsymbol{x},\boldsymbol{y})}{\sum_{k\in[K]}x^kc^k(x)},$$

where

$$f(\boldsymbol{x},\boldsymbol{y}) = \sum_{k\in[K]}\left[\left(x^kc^{k\prime}(x) + c^k(x) - c^k(y)\right)y^k - \left(x^k\right)^2c^{k\prime}(x)\right].$$

The proof goes as follows. We will first find an upper bound on $f(\boldsymbol{x},\boldsymbol{y})$ depending only on $\boldsymbol{x}$, Equation (6) below. Then, using the quantities $\delta(\boldsymbol{c})$ and $\lambda(\boldsymbol{c})$ defined in the statement of the proposition, we will get an upper bound on $\beta(\boldsymbol{c})$ expressed as a supremum of a concave function on a convex domain, Equation (7), which will lead by straightforward computations to the desired expression.

We compute now an upper bound on $f(\boldsymbol{x},\boldsymbol{y})$ depending only on $\boldsymbol{x}$. We have

$$f(\boldsymbol{x},\boldsymbol{y}) \leq \sup_{y\in\mathbb{R}_+,\ell\in[K]}\left(x^\ell c^{\ell\prime}(x) + c^\ell(x) - c^\ell(y)\right)y - \sum_{k\in[K]}\left(x^k\right)^2c^{k\prime}(x), \tag{4}$$

since for a $\boldsymbol{y}\in\mathbb{R}_+^K$ with fixed sum $y = \sum_{k\in[K]}y^k$, the first sum in the definition of f can be made maximum by putting all the weight on a single term.

The map $g : y \mapsto \left(x^\ell c^{\ell\prime}(x) + c^\ell(x) - c^\ell(y)\right) y$ is concave on $\mathbb{R}_+$. Its derivative is

$$g'(y) = x^\ell c^{\ell\prime}(x) + c^\ell(x) - c^\ell(y) - y c^{\ell\prime}(y).$$

Let $y^\star \in \mathbb{R}_+$ such that $g'(y^\star) = 0$. We have

$$y^\star c^{\ell\prime}(y^\star) + c^\ell(y^\star) = x^\ell c^{\ell\prime}(x) + c^\ell(x) \tag{5}$$

and thus, since $x^\ell \le x$,

$$x^\ell c^{\ell\prime}(x^\ell) + c^\ell(x^\ell) \le y^\star c^{\ell\prime}(y^\star) + c^\ell(y^\star) \le x c^{\ell\prime}(x) + c^\ell(x).$$

Since the map $u \mapsto u c^{\ell\prime}(u) + c^\ell(u)$ is nondecreasing, these inequalities imply

$$x^\ell \le y^\star \le x.$$

Hence

$$g(y^\star) = \left(x^\ell c^{\ell\prime}(x) + c^\ell(x) - c^\ell(y^\star)\right) y^\star \le (x^\ell + x - y^\star) c^{\ell\prime}(x) y^\star \le \left(\frac{x + x^\ell}{2}\right)^2 c^{\ell\prime}(x),$$

where the first inequality is a consequence of the convexity of $c^\ell(\cdot)$ and where the second is obtained via direct calculations. Using this bound in Equation (4), we get an upper bound that does not depend on $\boldsymbol{y}$ and which is valid for all $\boldsymbol{x} \in \mathbb{R}_+^K$:

$$\begin{aligned} f(\boldsymbol{x}, \boldsymbol{y}) &\le \sup_{\ell \in [K]} \left(\frac{x + x^\ell}{2}\right)^2 c^{\ell\prime}(x) - \sum_{k \in [K]} \left(x^k\right)^2 c^{k\prime}(x) \\ &\le \sup_{\ell \in [K]} \frac{x^2 c^{\ell\prime}(x)}{4} \left[\left(1 + 2\frac{x^\ell}{x} - 3\left(\frac{x^\ell}{x}\right)^2\right) - \frac{4}{\delta(\boldsymbol{c})} \sum_{k \neq \ell} \left(\frac{x^k}{x}\right)^2\right]. \end{aligned} \tag{6}$$

Using Equation (6) in the definition of $\beta(\boldsymbol{c})$ and with the help of the parameter $\lambda(\boldsymbol{c})$ defined in the statement of the proposition, we get

$$\beta(\boldsymbol{c}) \le \frac{\lambda(\boldsymbol{c})}{4} \sup_{\boldsymbol{x} \in \mathbb{R}_+^K, \ell \in [K]} \left[\left(1 + 2\frac{x^\ell}{x} - 3\left(\frac{x^\ell}{x}\right)^2\right) - \frac{4}{\delta(\boldsymbol{c})} \sum_{k \neq \ell} \left(\frac{x^k}{x}\right)^2\right].$$

Without loss of generality we can assume that the maximum is attained with $\ell = 1$. This inequality can thus be rewritten as

$$\beta(\boldsymbol{c}) \le \frac{\lambda(\boldsymbol{c})}{4} \sup_{\boldsymbol{z} \in \triangle} \left[\left(1 + 2z^1 - 3\left(z^1\right)^2\right) - \frac{4}{\delta(\boldsymbol{c})} \sum_{k=2}^{K} \left(z^k\right)^2\right], \tag{7}$$

where $\triangle$ is the $(K-1)$-dimensional simplex $\{\boldsymbol{z} \in \mathbb{R}_+^K : \sum_{k \in [K]} z^k = 1\}$. The value of right-hand side can be obtained by maximizing a concave function on a convex domain. We compute $\boldsymbol{z}^\star \in \triangle$ maximizing its value:

$$z^{\star,1} = \frac{1 + \theta(\boldsymbol{c})}{3 + \theta(\boldsymbol{c})} \quad \text{and} \quad z^{\star,2} = \cdots = z^{\star,K} = \frac{1 - z^{\star,1}}{K - 1},$$

where $\theta(\boldsymbol{c}) = \frac{4}{\delta(\boldsymbol{c})(K-1)}$. Inequality (7) becomes then

$$\beta(\boldsymbol{c}) \le \frac{\lambda(\boldsymbol{c})}{4} \left[1 + 2z^{\star,1} - 3(z^{\star,1})^2 - \theta(\boldsymbol{c})(1 - z^{\star,1})^2\right],$$

which leads to the upper bound given in the statement of the proposition. □

Note that the proof of Proposition 5 does not seem to be adaptable to deal with a parameter ω analogue to the one of Harks [6].

We explain now how to deduce Theorem 1 and Proposition 1 from Proposition 5.

Proof of Theorem 1. We have $\lambda(\mathcal{C})$ larger that any $\lambda(\boldsymbol{c})$, and we use $\dfrac{1}{1+\frac{4}{3}\frac{1}{\delta(\boldsymbol{c})(K-1)}} \leq 1$. □

Proof of Proposition 1. In this special case, we have Δ larger than any $\delta(\boldsymbol{c})$, and we have moreover $\delta(\boldsymbol{c}) = \lambda(\boldsymbol{c})$ for all $\boldsymbol{c}$. A straightforward calculation leads to the desired formula. □

4. The price of anarchy with affine cost functions is unbounded

In this section, we prove Proposition 2 by exhibiting an instance of the game with affine cost functions giving a price of anarchy which can be made arbitrarily large.

Consider the network with one origin-destination pair and two parallel arcs a and b. There are two players. The first player has a total demand of M and his cost functions are $c_a^1(x) = x$ and $c_b^1(x) = x + 2M$. The second player has a total demand of 1 and his cost functions are $c_a^2(x) = 2M^2x + 1$ and $c_b^2(x) = M^3x$. Denote by $I(M)$ this instance.

Since the network has parallel arcs, the Nash equilibrium is unique [9]. We prove that it is reached with the multiflow $\vec{\boldsymbol{x}}$ where player 1 puts all his demand on the arc a and player 2 puts all his demand on the arc b. Indeed, we have in this case, for any $\boldsymbol{y}^1 \in \mathcal{F}^1$ and $\boldsymbol{y}^2 \in \mathcal{F}^2$,

$$\tilde{c}_a^1(\boldsymbol{x}_a)(y_a^1 - x_a^1) + \tilde{c}_b^1(\boldsymbol{x}_b)(y_b^1 - x_b^1) = 2M(y_a^1 - M) + (2M+1)(M - y_a^1) = M - y_a^1 \geq 0$$
$$\text{and } \tilde{c}_a^2(\boldsymbol{x}_a)(y_a^2 - x_a^2) + \tilde{c}_b^2(\boldsymbol{x}_b)(y_b^2 - x_b^2) = (2M^3+1)(1 - y_b^2) + 2M^3(y_b^2 - 1) = 1 - y_b^2 \geq 0.$$

Proposition 3 gives then that $\vec{\boldsymbol{x}}$ is a Nash equilibrium. The social cost at equilibrium is $C(\vec{\boldsymbol{x}}) = M^2 + M^3$.

Consider now the multiflow $\vec{\boldsymbol{z}}$ where player 1 puts all his demand on the arc b and player 2 puts all his demand on the arc a. These flows are feasible and give a social cost $C(\vec{\boldsymbol{z}}) = 5M^2 + 1$. We have then

$$\mathrm{PoA}(I(M)) \geq \frac{C(\vec{\boldsymbol{x}})}{C(\vec{\boldsymbol{z}})} = \frac{M^3 + M^2}{5M^2 + 1}.$$

Since $\lim_{M\to+\infty} \mathrm{PoA}(I(M)) = +\infty$, we get the result.

Proof of Proposition 2. The instance $I(6M)$ gives a price of anarchy greater than M. □

5. Discussion and open questions

The bound of Proposition 5 makes sense only when $\lambda(\boldsymbol{c}) - 3 < \frac{4}{\delta(\boldsymbol{c})(K-1)}$, since otherwise it is larger than 1. When $\lambda(\boldsymbol{c}) > 3$, this condition is met only when

$$K < 1 + \frac{4}{\delta(\boldsymbol{c})(\lambda(\boldsymbol{c}) - 3)}.$$

In other words, there is a whole range of cost functions and numbers of players for which we are unable to provide any concrete bounds. It would be interesting to extend the bound to a larger set of instances. Proposition 2 shows that the price of anarchy is unbounded as soon as the set of affine cost functions is included in the set of allowable cost functions. Its proof needs that the demand of one player become infinitely larger than the demands of the others. The question whether the price of anarchy is bounded with affine cost functions, when for instance the quantity $\frac{d^j}{\sum_{k\in[K]} d^k}$ remains bounded for each player j, remains an open question.

Another result from Cominetti et al. [3] that can be extended to the multiclass case is their Proposition 3.7 stating that the social cost at equilibrium is bounded by the optimal cost of the game where the demands are multiplied by $1+\beta(\mathcal{C})$. More precisely, we can adapt their proof to find the following proposition.

Proposition 6. *Consider an atomic splittable network congestion game with player-specific cost functions. Suppose that the available cost functions in $\mathcal{C}$ are differentiable, nonnegative, increasing, and convex. Consider an instance I with an equilibrium multiflow $\vec{\boldsymbol{x}}^{NE}(I)$ and the instance αI where all demands are multiplied by $\alpha \geq 1$, with an optimal multiflow $\vec{\boldsymbol{x}}^{OPT}(\alpha I)$. Suppose that $\beta(\mathcal{C}) < 1$, we have*

$$C(\vec{\boldsymbol{x}}^{NE}(I)) \leq \frac{1}{\alpha - \beta(\mathcal{C})} C(\vec{\boldsymbol{x}}^{OPT}(\alpha I)).$$

Proof. The proof is almost the same as in Cominetti et al. [3]. For the ease of reading, we denote $\vec{\boldsymbol{x}}^{NE} = \vec{\boldsymbol{x}}^{NE}(I)$. Let $\vec{\boldsymbol{y}}$ be a flow feasible for the instance αI, then

$$\begin{aligned}
\alpha C(\vec{\boldsymbol{x}}^{NE}) &= \alpha \left[\sum_{k\in[K]} \sum_{a\in A} \left(c_a^k(x_a^{NE}) - \tilde{c}_a^k(\boldsymbol{x}_a^{NE}) \right) x_a^{NE,k} + \tilde{c}_a^k(\boldsymbol{x}_a^{NE}) x_a^{NE,k} \right] \\
&\leq \alpha \left[\sum_{k\in[K]} \sum_{a\in A} \left(c_a^k(x_a^{NE}) - \tilde{c}_a^k(\boldsymbol{x}_a^{NE}) \right) x_a^{NE,k} + \tilde{c}_a^k(\boldsymbol{x}_a^{NE}) \frac{y_a^k}{\alpha} \right] \\
&\leq \sum_{k\in[K]} \sum_{a\in A} \left(c_a^k(x_a^{NE}) - \tilde{c}_a^k(\boldsymbol{x}_a^{NE}) \right) x_a^{NE,k} + \tilde{c}_a^k(\boldsymbol{x}_a^{NE}) y_a^k \\
&\leq \beta(\mathcal{C}) C(\vec{\boldsymbol{x}}^{NE}) + C(\vec{\boldsymbol{y}})
\end{aligned}$$

where we use Equation (2) with $\frac{y_a^k}{\alpha}$ to get the first inequality, the fact that $\alpha \geq 1$ and $c_a^k(x_a^{NE}) - \tilde{c}_a^k(\boldsymbol{x}_a^{NE}) \leq 0$ to get the second inequality, and the definition of $\beta(\cdot)$ to get the last inequality.

We finish by taking $\vec{\boldsymbol{y}} = \vec{\boldsymbol{x}}^{OPT}(\alpha I)$. $\square$

This proposition extends Proposition 4 which deals with the case $\alpha = 1$. In particular we have $C(\vec{\boldsymbol{x}}^{NE}(I)) \leq C(\vec{\boldsymbol{x}}^{OPT}((1+\beta(\mathcal{C}))I))$.

Another possible further development would be to compare the game studied in this paper with the nonatomic case. Cominetti et al. [3] proved that when all players have same cost functions, same demand, and same origin-destination pair, the social cost at equilibrium of the atomic game is bounded by the one of the corresponding nonatomic game. In particular, the price of anarchy of the atomic game is bounded by the one of the nonatomic game. The key point in their context is that the atomic game is potential, which is unlikely to be the case here. Whether this bound holds with player-specific cost functions is an open question.

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